Handling Equality Constraints in Robust Design Optimization

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Robust design optimization (RDO) is a powerful tool for managing the tradeoffs between optimal performance and performance stability. A robust design is one where system performance remains relatively unchanged (stable) when exposed to stochastic conditions. Although many advances in the area of RDO have been realized over the past decade, most RDO methods do not address a critical branch of design optimization – namely, equality constrained problems. In this paper, we offer a new perspective on handling equality constraints in RDO. As part of the development, we identify two types of equality constraints; those that must be strictly satisfied – regardless of stochastic conditions, and those that cannot be strictly satisfied because of stochastic conditions. Each constraint type is mathematically defined, which allows for the process of classifying any given equality constraint. We also provide a generic means for handling both equality constraint types in RDO. A simple structural optimization problem is used to illustrate our approach.

1. Introduction and Literature Survey

Uncertain design information can significantly impact the success of the engineering design process. Such uncertainty – which is always present – comes in numerous forms, which include imprecise material properties, imperfect manufacturing methods, unknown loading conditions, and over simplified engineering models. When these uncertainties are not considered during the optimization of a design, the obtained solution is likely to be more “high risk” than “optimal”. For this reason, properly handling uncertainty is a critical element of design optimization. Robust design optimization (RDO) specifically seeks to account for design uncertainties and to reduce their negative effect.

During the past two decades, RDO has become a significant area of research in the design community. The appeal of RDO is that its solution (a robust design) remains relatively unchanged (stable) when exposed to uncertain conditions. The need for RDO originates from the fact that uncertain conditions are always present. Although many advances in the area of RDO have been realized, most RDO methods do not address a critical branch of design optimization – namely, equality constrained problems. In this paper we provide a new perspective on handling equality constraints in RDO.

Literature Survey

In the following paragraphs, we present a brief survey of the RDO literature with a focus on how various RDO methods deal with constraint satisfaction. In the early 1980’s, the first RDO approaches focused primarily on inequality constraint satisfaction. Balling et al., proposed a worst-case tolerance approach to robust design. They consider worst case tolerances and shift optimization constraints such that all possible combinations of worst case tolerances result in a feasible design. Interestingly, these authors do not address the equality constrained problem.

Parkinson et al., develop a method for robust design optimization called feasibility robustness. The main elements of their approach are (i) to maintain design feasibility given input variations, and (ii) to minimize the effect of variations on the performance of the system. A sensitivity model is used to account for variations in the constraint functions; however equality constraints are not considered.

Otto and Antonsson expand Taguchi’s robust design approach by including design constraints using constrained optimization methods. They directly address RDO with inequality constraints, but do not provide explicit details related to equality constrained problems. Interestingly, they say that “equality constraints are not discussed [here], but are easily incorporated.” In contrast, Ramakrishnan and Rao indicate that equality constraints are generally difficult to handle in RDO. In their work, they start with a conventional (non-robust) optimization problem that includes both inequality and equality constraints. They address the equality constrained problem by adding new constraints that are obtained by relaxing the equality constraints. According to the authors, this is done to make the non-linear optimization problem tractable and easier to solve.

The past decade has seen continued successful efforts to improve RDO approaches – but with little emphasis on handling equality constrained problems. Similar to
the early developments in robust design optimization, numerous recent publications do not provide specific details for handling equality constraints in RDO.\textsuperscript{1,8–16}

Most RDO approaches start by converting a conventional (deterministic) design optimization (CDO) problem into a RDO problem where objectives have been added and/or constraints have been modified to account for uncertainties. In this paper, we too consider the important task of CDO to RDO conversion. We note that one of the greatest challenges of this conversion is to ensure that we have not inadvertently changed the problem we wish to solve during the conversion process. It is our tenet that, although many approaches do not attempt to do so, it is crucial that during this conversion some of the equality constraints be retained in their exact and strict form. Anything short of this retention could be a significant departure from the original CDO problem being solved.

We believe that as advances in RDO methods continue to come forth, it is critical that problems with equality constraints be adequately addressed. Any other approach should be considered incomplete and potentially incorrect.

A limited number of publications provide details for handling equality constraints in RDO problems. In these publications, three distinct and conflicting approaches for handling equality constraints can be found. They are: (a) to relax the equality constraint, (b) to satisfy the equality constraint in a probabilistic sense, and (c) to remove the equality constraint through substitution. Each is briefly discussed.

Su and Renaud,\textsuperscript{17} Ramakrishanah and Rao,\textsuperscript{7} Fares et al.,\textsuperscript{18} and Messac and Ismail-Yahaya,\textsuperscript{19} all take an equality constraint relaxation approach. Su and Renaud\textsuperscript{17} indicate that it is nearly impossible to satisfy equality constraints in RDO, and that in order to solve such problems, equality constraints must first be relaxed. Ramakrishanah and Rao\textsuperscript{7} minimize the variation of the system performance using techniques of stochastic non-linear programming. They use a Taylor series expansion of the objective function about the mean values of the design variables. This results in the expected value of the objective function. Similarly, the inequality and equality constraints are expanded about the mean values of the design variables.

Fares et. al.,\textsuperscript{18} convert a deterministic linear programming formulation with equality constraints into another linear programming formulation with slack variables that augment the equality constraint. Messac and Ismail-Yahaya\textsuperscript{19} suggest that equality constraints theoretically leave no room for the flexibility inherent in non-deterministic problems. Furthermore, they suggest that some compromise must take place. Each equality constraint is then converted into two inequality constraints.

Others have handled equality constraints in RDO by satisfying the constraint in a probabilistic sense. Sundaresan, et al.,\textsuperscript{20} and Putko, et al.,\textsuperscript{21} use this approach. Satisfaction of the equality constraint is only enforced at the “target” design – or the expected value of the design parameters. These authors do indicate, however, that these constraints may be violated when exposed to uncertainties. Sundaresan, et al.,\textsuperscript{20} define five classes of inequality constraint violation. Only one class is given for equality constraints – the violated class, since any variation in the design causes the equality constraint to be violated.

In all of the cases described above, it has been emphasized that equality constraints cannot be satisfied under stochastic conditions. Das\textsuperscript{22} supports a slightly different idea. He suggests that there is a special type of equality constraint that must be satisfied; such as a physics based equality constraint. To ensure satisfaction of this special equality constraint, Das substitutes the equality constraint back into the objective function, which converts the equality constrained RDO problem into an unconstrained RDO problem.

Observations from Survey

We make the following observations from our survey of the RDO literature. Observation 1: Most of the literature does not address the handling of equality constraints in RDO problems. Observation 2: The limited number of publications that do address equality constrained problems separately provide three distinct approaches for handling the equality constraint. They are (a) to relax the equality constraint, (b) to satisfy the equality constraint in a probabilistic sense, and (c) to remove the equality constraint through substitution. Observation 3: These three basic approaches for handling equality constraints in RDO are conflicting in principle.

Such conflicts in proposed methods for handling equality constraints lead to the research question addressed in this paper: What are the conditions that govern the strictness to which equality constraints should be enforced in RDO? In this paper, we seek to answer this question by providing a new perspective on handling equality constraints in RDO. As part of the development, we identify two types of equality constraints; those that must be strictly satisfied – regardless of stochastic conditions, and those that cannot be strictly satisfied because of stochastic conditions. Each constraint type is mathematically defined, which allows for the process of classifying any given equality constraint. We also provide a generic means for handling both equality constraint types in RDO.

The remainder of this paper is presented as follows. In Section 2, we provide an analytical development for solving the RDO problem – with explicit details regarding the handling of equality constraints. In Section 3, we make important observations about the analytical development presented in Section 2, and about RDO in general. Section 4 presents a simple
structural optimization example that illustrates our approach, and concluding remarks are given in Section 5.

2. Analytical Development: Equality Constraint Handling in RDO

In this section, we analytically develop an approach for handling equality constraints in robust design optimization (RDO). As discussed in the introduction, we seek to account for uncertainty during the optimization process; otherwise, the final design might be at risk of failure when exposed to uncertainties.

The analytical development presented in this section starts at the root objective of RDO, which is to optimize a design while accounting for uncertainties. That is, we wish to solve the following conventional (non-robust) optimization problem when the parameters \( x \) and \( p \) are stochastic.

**Problem 1: Conventional Design Optimization (CDO)**

\[
\begin{align*}
\min_x f(x, p) \quad (1) \\
\text{subject to} \\
g_k(x, p) &\leq 0 \quad (k = 1, 2, ..., n_g) \quad (2) \\
h_k(x, p) & = 0 \quad (k = 1, 2, ..., n_h) \quad (3) \\
x_{i_{\text{min}}} \leq x_i \leq x_{i_{\text{max}}} \quad (i = 1, 2, ..., n_x) \quad (4)
\end{align*}
\]

where \( x \) is a vector of design parameters that are actively changed during the optimization process, and \( p \) is a vector of fixed (constant valued) parameters. The number of inequality constraints, equality constraints, and design parameters are denoted as \( n_g \), \( n_h \), and \( n_x \), respectively.

Numerous approaches have been developed that seek to solve this optimization problem; a few of them are discussed in our literature survey. As part of this development, we provide a new prospective on solving Problem 1 under stochastic conditions. We begin the development by examining the objective function and the inequality constraints and make appropriate changes to them that account for uncertainty. A framework for handling the equality constraints is then developed. Our development proceeds based on the following assumptions.

Assumptions:

1. All \( x \) and \( p \) are independent parameters.

2. The stochastic natures of \( x \) and \( p \) are characterized by prescribed variations with rectangular distributions of the form shown in Fig. 1. The parameter \( x \) is along the horizontal axis and the probability of \( x \) is on the vertical axis. The mean value of \( x \) is denoted by \( \bar{x} \), and the maximum variation thereof by the tilde.

![Rectangular Probability Distribution](image)

**Fig. 1 Rectangular Probability Distribution**

3. The maximum variations of \( x \) and \( p \), denoted as \( \tilde{x} \) and \( \tilde{p} \), are small valued (so as to allow for linear approximations).

4. The functions \( f(x, p) \), \( g(x, p) \), and \( h(x, p) \) are differentiable.

5. When accounting for the stochastic nature of \( x \) and \( p \), we wish to optimize the original function of Problem 1 and its variation.

We note that the assumed rectangular probability distribution is only invoked for simplicity of presentation. The study of more realistic distributions would be an important next step.

Based on Assumptions 2 and 5, the objective function (Eq. 1) can be transformed to the following equation when the stochastic natures of \( x \) and \( p \) are considered in the optimization problem.

\[
\min_x J = f(x, p) + \alpha \tilde{f}(x, p, \tilde{x}, \tilde{p}) \quad (5)
\]

The tilde over the variable represents its variations, and \( \alpha \) is a scalar weight. Based on Assumptions 1–4, we can use a first-order Taylor series expansion about \( \bar{x} \) and \( \bar{p} \) to determine the variations transmitted from the stochastic parameters to the functions \( f(x, p) \), \( g(x, p) \), and \( h(x, p) \). The first-order Taylor series expansion leads to the following expression for the variation of \( f(x, p) \). For notation simplicity, we define \( \tilde{f} = \tilde{f}(x, p, \tilde{x}, \tilde{p}) \).

\[
\tilde{f} = \sum_{i=1}^{n_x} \left( \frac{\partial f(x, p)}{\partial x_i} \right)^2 \tilde{x}_i + \sum_{i=1}^{n_p} \left( \frac{\partial f(x, p)}{\partial p_i} \right)^2 \tilde{p}_i \quad (6)
\]

To ensure design feasibility under stochastic conditions, we can shift (make more stringent) the inequality constraints in Problem 1 so that given worst-case variations, \( \tilde{x} \) and \( \tilde{p} \), the constraint is still satisfied. With this shift, Eq. 2 becomes

\[
g_k(x, p) + \tilde{g}_k(x, p, \tilde{x}, \tilde{p}) \leq 0 \quad (k = 1, 2, ..., n_g) \quad (7)
\]

where \( \tilde{g}_k(x, p, \tilde{x}, \tilde{p}) \) is found using a first-order Taylor series expansion as discussed for the case of \( f(x, p) \) above. When in the form of Eq. 6, we note that \( \tilde{g}_k(x, p, \tilde{x}, \tilde{p}) \) is conservative. Likewise, to ensure design feasibility we can shift the side constraints (Eq.
Type 1 Equality Constraint: A Type 1 equality constraint is a constitutive equality relationship that is strictly satisfied – regardless of all stochastic conditions. Type 1 equality constraints are denoted as $h^{T1}$.

Type 2 Equality Constraint: A Type 2 equality constraint is an equality relationship that cannot be strictly satisfied due to stochastic conditions. Type 2 equality constraints are denoted as $h^{T2}$.

To exemplify and further explore these two equality constraint types, let us consider the design of a three-piece modular bridge shown spanning an open channel in Fig. 3. The length of each modular section is denoted as $x_i$ for $i = 1, 2, 3$. The total length of the bridge is indicated as $L$ and the width of the channel is given as $L_c$. Let us assume that the following equality constraints are part of a deterministic optimization bridge problem.

\[
L - x_1 - x_2 - x_3 = 0 \tag{9}
\]

\[
\varphi L_c - x_1 - x_2 - x_3 = 0 \tag{10}
\]

where $\varphi$ is a scalar parameter with a value greater than 1. In this hypothetical example, Eq. 9 serves to keep the modular sections together at the joints during the optimization process, and Eq. 10 ensures that the bridge is sufficiently long. Importantly, we note that Eqs. 9 and 10 are fundamentally different, and require fundamentally different approaches for satisfying them under stochastic conditions. Each of these constraints is further discussed in the following.

Assuming that $x_i$ is stochastic and satisfies Assumptions 1–3, and that $L$ is not an independent parameter (rather it is a function of $x_i$), we make an important observation about Eq. 9. Regardless of how the lengths $x_i$ vary, the total length of the bridge will always be $L = x_1 + x_2 + x_3$. That is, Eq. 9 is always satisfied – regardless of all stochastic conditions. Therefore by definition, Eq. 9 is a Type 1 equality constraint. As a note, Type 1 equality constraints are also often included in the optimization problem statement to ensure that the optimal solution satisfies the physical laws of nature. Examples of Type 1 equality constraints include $F = ma$, and force and moment equilibrium equations.

If by definition, Type 1 equality constraints are strictly satisfied regardless of the stochastic nature of $x$ and $p$, then one of two things must be true. Either $h^{T1}$ is not a function of $x$ or $p$, and therefore its variation due to $\hat{x}$ and $\hat{p}$ is zero. Or, $h^{T1}$ is a function of $x$ or $p$ such that, whatever variations of $x$ and $p$ exist, the net change in $h^{T1}$ is zero. In both cases the
The following property is true.

\[ \Delta h^{T1}(x, p, \Delta x, \Delta p) = 0 \]  

(11)

where \( \Delta h^{T1}(x, p, \Delta x, \Delta p) \) is the actual variation of the constraint \( h^{T1} \) when \( x \) and \( p \) are stochastic, and where \( \Delta x \) is any variation of \( x \) and \( \Delta p \) is any variation of \( p \) – be it the maximum variations \((\bar{x}, \bar{p})\) or not. For notation simplicity, we define \( \Delta h^{T1} = \Delta h^{T1}(x, p, \Delta x, \Delta p) \).

The actual constraint variation can be written as

\[ \Delta h^{T1} = h^{T1}(x + \Delta x, p + \Delta p) - h^{T1}(x, p) \]  

(12)

A key observation in the modular bridge example (Eq. 9) is that although \( x_i \) are independent parameters, \( L \) is NOT independent of \( x_i \). Mutual dependence between terms is another characteristic of Type 1 equality constraints.

Now let us consider the equality constraint given in Eq. 10. Equation 10 is fundamentally different than Eq. 9 because there is no mutual dependence between terms. The term \( \varphi L_e \) is an independent parameter, which is also used in other relationships governing the optimization problem. As such, the variations in \( x_1 \) are not guaranteed or even likely to be cancelled by equal and opposite variations in \( x_2 \) or \( x_3 \). Therefore, it cannot be strictly satisfied when uncertainties are included as part of the optimization. Therefore Eq. 10 is a Type 2 equality constraint. Mathematically, Type 2 equality constraints are defined as having the following property:

\[ \Delta h^{T2}(x, p, \Delta x, \Delta p) \neq 0 \]  

(13)

where \( \Delta h^{T2} \) is the actual variation of the constraint \( h^{T2} \), and is of the same form given in Eq. 12.

Given the existence of at least Type 1 and Type 2 equality constraints, we return to Problem 1 and our main RDO objective, which was to solve the optimization problem while including the effects of uncertainty. If Eq. 3 of Problem 1 is a Type 1 equality constraint, then it must be strictly satisfied, therefore it stays in its original form when uncertainties are considered. That is,

\[ h_k^{T1}(x, p) = 0 \quad (k = 1, 2, ..., n_{hT1}) \]  

(14)

If Eq. 3 is a Type 2 equality constraint, then there are two possibilities for handling it under uncertainty; (i) the constraint can be relaxed (made less stringent), or (ii) the constraint can be satisfied at the mean parameter values only. Type 2 equality constraints that are handled using the former are denoted as \( h_k^{T2r} \) and referred to as Type 2r, and those using the latter are denoted as \( h_k^{T2m} \) and referred to as Type 2m. The decision to model Type 2 equality constraints by relaxing or satisfying them at the mean should be made after assessing why the given constraint is included in the original problem. For example, if a market study showed that the ideal surface area of desk is 2 m² then perhaps it can be concluded that satisfying this constraint only that the mean parameter values is sufficient – since small variations in the actual surface area are not likely to negatively impact the success of the product.

When Eq. 3 is relaxed, it can be transformed into two inequality constraints of the following form

\[ h_k^{T2r}(x, p) + |\Delta h_k^{T2r}(x, p, \Delta x, \Delta p)| \geq 0 \]  

(15)

\[ h_k^{T2r}(x, p) - |\Delta h_k^{T2r}(x, p, \Delta x, \Delta p)| \leq 0 \]  

(16)

where \( k = 1, ..., n_{hT2r} \). When Eq. 3 is to be satisfied only at the mean value of the parameters, then it is written as follows

\[ h_k^{T2m}(x, p) = 0 \]  

(17)

where \( x \) and \( p \) are the mean parameter values, and \( k = 1, ..., n_{hT2m} \). Satisfying Type 2 equality constraints at the mean parameter values only does not imply that \( \Delta h^{T2}(x, p, \Delta x, \Delta p) = 0 \). Rather it means that the designer declares \( \Delta h^{T2} \) to be of little consequence.

We note that the notion of satisfying the constraint only at the mean parameter values stems from the work of Sundaresan, et al.,\(^{20}\) and Putko, et al.,\(^{21}\) which was the approach they used to handle equality constraints under uncertainty.

With Eqs. 5–17 and the related discussion, we have shown how Problem 1 can be solved when the stochastic nature of \( x \) and \( p \) are included in the optimization. Below in Problem 2, the main equations from the preceding discussion are rewritten as constituent components of the RDO problem statement.

**Problem 2: Robust Design Optimization (RDO)**

\[ \min_x J = f(x, p) + \alpha \tilde{f}(x, p, \bar{x}, \bar{p}) \]  

(18)

subject to

\[ g_k(x, p) + \tilde{g}_k(x, p, \bar{x}, \bar{p}) \leq 0 \quad (k = 1, 2, ..., n_g) \]  

(19)

\[ h_k^{T1}(x, p) = 0 \quad (k = 1, 2, ..., n_{hT1}) \]  

(20)

\[ h_k^{T2m}(x, p) = 0 \quad (k = 1, 2, ..., n_{hT2m}) \]  

(21)

\[ h_k^{T2r}(x, p) + |\Delta h_k^{T2r}(x, p, \Delta x, \Delta p)| \geq 0 \]  

(22)

\[ h_k^{T2r}(x, p) - |\Delta h_k^{T2r}(x, p, \Delta x, \Delta p)| \leq 0 \]  

(23)

\[ x_{i_{\text{min}}} + \bar{x}_i \leq x_i \leq x_{i_{\text{max}}} - \bar{x}_i \quad (i = 1, 2, ..., n_x) \]  

(24)}
3. Observations and Discussion

In this section, we make important observations about the analytical development presented in Section 2, and comment on pertinent aspects of robust design optimization (RDO). This section is divided into three small subsections, which discuss the following topics. (1) A single approach for handling both Type 1 and Type 2\(r\) equality constraints. (2) Prescribed versus non-prescribed variations, and (3) formulating the aggregate objective function in RDO.

**A Single Approach for Handling Equalities**

We make an interesting observation regarding the analytical development presented in Section 2. Because Type 1 equality constraints are such that \(\Delta h^{\text{T1}}(x, p, \Delta x, \Delta p) = 0\), a single computational approach can be used to handle both Type 1 and Type 2\(r\) equality constraints under stochastic conditions. That is, for Type 1 and Type 2\(r\) equality constraints,

\[
h_k(x, p) + |\Delta h_k(x, p, \Delta x, \Delta p)| \geq 0 \quad (25)
\]

\[
h_k(x, p) - |\Delta h_k(x, p, \Delta x, \Delta p)| \leq 0 \quad (26)
\]

for \((k = 1, 2, ..., n_{h_{T1}} + n_{h_{T2r}})\). Given that the variation of \(h^{\text{T1}}\) is zero \((\Delta h^{\text{T1}}(x, p, \Delta x, \Delta p) = 0)\), Eqs. 25 and 26 collapse to the following for Type 1 equality constraints.

\[
h_k^{\text{T1}}(x, p) = 0 \quad (k = 1, 2, ..., n_{h_{T1}}) \quad (27)
\]

The collapsing of Eqs. 25 and 26 to Eq. 27 for Type 1 equality constraints makes this single approach equivalent to the approach developed in Section 2. Using the approach discussed in the current section eliminates the need to identify the constraint type, which may be particularly useful for some complex problems. Even so, we advocate identifying the equality constraint type and treating each type as discussed in the analytical development of Section 2. Doing so allows the designer to better understand the optimization problem being solved, and reduces unneeded function evaluations.

**Prescribed versus Non-Prescribed Variations**

In the analytical development of Section 2, we assumed that the stochastic natures of \(x\) and \(p\) where characterized by prescribed variations. Although prescribed variations represent an important and practical branch of RDO problems, we note that under many other practical circumstances, the variations are not prescribed; further, the designer is often required to specify them. In such cases, optimal variations for the parameters are sought during the optimization process. The important related point is that allowable variation levels significantly impact cost.

We now restate Problem 2 for non-prescribed variations in the parameters. For notation simplicity, we let \(f = f(x, p)\) and \(\tilde{f} = f(x, p, \tilde{x}, \tilde{p})\).

**Problem 3: RDO with Non-Prescribed Variations**

\[
\min J = f + \alpha \tilde{f} + \Psi(x, p, \tilde{x}, \tilde{p}) + \gamma \tilde{\Delta}^{2r} \quad (28)
\]

subject to Eqs. 19–24. When comparing Problem 3 to Problem 2, it can be seen that the third and fourth terms of the aggregate objective function have been added and that the optimization is now over \(x, \tilde{x}\), and \(\tilde{p}\). The third term of Eq. 28 is a function that accounts for the cost of decreasing parameter variation (or tightening manufacturing tolerances). The fourth term is critical in that it keeps Type 2\(r\) equality constraints from being excessively relaxed. The variable \(\gamma\) is a scalar weight. Messac and Ismail-Yahaya\(^{19}\) offer more on the topic of non-prescribed variations in RDO, including the combination of prescribed and non-prescribed variations.

**The RDO Aggregate Objective Function**

We now comment on the important task of formulating the RDO aggregate objective function for multiobjective optimization. For simplicity we discuss the prescribed variations case, although the same basic principle applies to the case with non-prescribed variations. A deterministic multiobjective optimization problem of \(n_m\) metrics may result in a RDO problem with \(2 \times n_m\) metrics – one metric for each of the original metrics, and one for each of the variations thereof.

Whatever approach is used to formulate the aggregate objective function for the \(2 \times n_m\) metrics, it is important that no particular set of metrics becomes prematurely combined into small sub-objectives such as “minimize variation”, which is one approach typically found in the literature. Importantly, the designer should formulate the aggregate objective function so that he or she can have adequate control over changing the degree to which each individual metric (including individual variations) should be optimized. The premise for this is that one may wish to significantly minimize the variation of say metric 1, while at the same time have little desire to minimize the variation of metric 2. The approach used in Eq. 5 in fact suffers from this deficiency. In addition, the preference embodied in the original aggregate objective function does not necessarily translate to preference for the variations of the objectives. For example, one may wish to significantly minimize say objective 7, with little desire to minimize its variation. This issue is addressed by Messac and Ismail-Yahaya\(^{19}\) where a physical programming based approach to formulate the RDO aggregate objective function is developed.

4. A Simple Structural Example

In this section, a simple two-bar truss is used to illustrate the developments presented in this paper. We present the example by first providing two equivalent deterministic optimization problem statements for the truss – one with equality constraints, and one without.
Design Metrics

Two Equivalent Deterministic Formulations for the Truss Optimization Problem

In this section, we provide two equivalent optimization formulations for optimizing the truss shown in Figure 4. Note that the equivalent formulations are for the deterministic case, where uncertainties are not considered. As part of the optimization problem we seek to minimize the squared nodal deflection at point \( P \) and minimize the total structural volume, subject to the normal stress and beam cross-sectional areas being within acceptable limits. The structure is exposed to constant horizontal and vertical loads as indicated in the figure. The following detailed truss information is used throughout the entire example, unless otherwise noted. We will subsequently refer to the following information as the “detailed truss information”.

Design Parameters

- \( a_1 \): cross-sectional area of bar 1 (m\(^2\))
- \( a_2 \): cross-sectional area of bar 2 (m\(^2\))
- \( b \): horizontal distance from the left most part of the truss to node \( P \) (m)

Constant Parameters

- \( L \): height of structure; 18,288 m
- \( W_1 \): horizontal load; \( 4.45 \times 10^5 \) N
- \( W_2 \): vertical load; \( 4.45 \times 10^6 \) N
- \( E \): modulus of elasticity; \( 1.99 \times 10^{11} \) Pa
- \( S_{max} \): maximum allowable stress; \( 3.79 \times 10^9 \) Pa
- \( a_{i_{min}} \): lower bound for \( a_i \); \( 5.16 \times 10^{-4} \) m\(^2\) (\( i = 1, 2 \))
- \( a_{i_{max}} \): upper bound for \( a_i \); \( 1.94 \times 10^{-3} \) m\(^2\) (\( i = 1, 2 \))
- \( b_{min} \): lower bound for \( b \); 9.144 m
- \( b_{max} \): upper bound for \( b \); 27,432 m
- \( w_1 \): scalar weight; 0.3
- \( w_2 \): scalar weight; 0.7

Truss Formulation 1: No Equality Constraints

\[
\min_{a, b} J = w_1 f_1 + w_2 f_2
\]

subject to

\[
S_i \leq S_{max} \quad (i = 1, 2)
\]
\[
a_{i_{min}} \leq a_i \leq a_{i_{max}} \quad (i = 1, 2)
\]
\[
b_{min} \leq b \leq b_{max}
\]

where the normal stress in each bar is denoted as \( S_i \).

We now consider the second truss optimization formulation. For this formulation, the information given as “detailed truss information” is used with the exception of the following additional information.

Additional Parameters for Truss Formulation 2

- \( u_1 \): horizontal deflection at node \( P \) (m)
- \( u_2 \): vertical deflection at node \( P \) (m)
- \( \theta \): acute angle between the horizontal and bar 1
- \( \beta \): acute angle between the horizontal and bar 2

Importantly, we note that the additional parameters are not independent. That is, each is a function of the original design parameters. We will show shortly the effect of such dependencies.

Truss Formulation 2: With Equality Constraints

\[
\min_{a, b, u, b, \beta} J = w_1 f_1 + w_2 f_2
\]

subject to

\[
S_i \leq S_{max} \quad (i = 1, 2)
\]
\[
a_{i_{min}} \leq a_i \leq a_{i_{max}} \quad (i = 1, 2)
\]
\[
b_{min} \leq b \leq b_{max}
\]
\[
W_1 = F_i \cos (\theta) - F_2 \cos (\beta)
\]
\[
W_2 = F_i \sin (\theta) + F_2 \sin (\beta)
\]
\[
\theta = \arctan (L/b)
\]
\[
\beta = \arctan (L/(2L - b))
\]

where the normal stress in each bar is denoted as \( S_i \), and \( F_i \) is the normal forces acting in bar \( i \) for \( i = 1, 2 \).
Equations 37 and 38 are force equilibrium constraints that keep the structure in static equilibrium. Equations 39 and 40 keep the structure connected a node $P$.

Both Truss Formulation 1 and Truss Formulation 2 yield the same optimal solution. The results for this deterministic case are shown in Table 1 and in Fig. 5. In the figure the optimal solution for both cases is shown as a triangle, and the Pareto frontier is shown as a solid curve. We note that the difference between Truss Formulation 1 and Truss Formulation 2 is that Eqs. 37–40 have been substituted into the objective function of Truss Formulation 1 (Eq. 29).

Optimizing the Truss under Stochastic Conditions

Recall that our main objective is to solve the optimization problem under stochastic conditions. Therefore, we use the developments presented in this paper to restate the truss formulations while accounting for uncertainties. As part of the RDO problem we wish to minimize the squared nodal deflection at node $P$, the structural volume, the variation of the deflection, and the variation of the volume, subject to the constraints of the original problem.

To avoid a cumbersome example, we will present details on solving Truss Formulation 2 (which is the truss formulation with equality constraints) under stochastic conditions, while we simply use the solution to the RDO problem stemming from Truss Formulation 1 as a reference. The detailed truss information given above is used again here with the following added information.

Additional Parameters for RDO Truss Formulation 2

$\tilde{a}_1$ variation of $a_1$: $1.00 \times 10^{-4}$ (m$^2$)

$\tilde{a}_2$ variation of $a_2$: $1.00 \times 10^{-4}$ (m$^2$)

$\tilde{b}$ variation of $b$: 1.00 (m)

$w_3$ scalar weight; 4

$w_4$ scalar weight; 181

Additional Metrics for RDO Truss Formulation 2

$\tilde{f}_1$ variation of squared nodal displacement at node $P$ (m$^2$)

$\tilde{f}_2$ variation of structural volume (m$^3$)

Revisiting the taxonomy chart given in Fig. 2, we can see that the inequality constraints in Eqs. 34–36 can be shifted so that given worst case variation, the constraints will still be satisfied. With such a shifting, the inequality constraints are written as shown below in Eqs. 42–44.

Figure 5 Optimal Solutions for Truss Example

Before being able to properly handle the equality constraints (Eqs. 37–40) under stochastic conditions, we need to identify which type of equality constraint they are. Using the mathematical definitions of Type 1 and Type 2 equality constraints, as given in Section 2, we can see that $\Delta h = 0$ for each equality constraint in this problem. Therefore all four of these constraints are Type 1 equality constraints. Based on the developments provided in Section 2, these constraints are to be strictly satisfied in the RDO problem. Equations 45–48 show that these Type 1 equality constraints are kept in their exact form when uncertainties are considered. We note that this particular example does not have Type 2 equality constraints. Had there been Type 2 equality constraints, we would have chosen to either relax those equality constraints, or satisfy them only at the mean parameter values as discussed in Section 2.

The optimization problem statement for Truss Formulation 2 is now given as follows when uncertainties are considered during the optimization process.

**RDO Truss Formulation 2: With Equality Constraints**

$$
\min_{a,b,u,\theta,\beta} \quad J = w_1 f_1 + w_2 f_2 + w_3 \tilde{f}_1 + w_4 \tilde{f}_2
$$

subject to

$$
S_i + \tilde{S}_i \leq S_{\max} \quad (i = 1, 2) \quad (42)
$$

$$
a_{i_{\min}} + \tilde{a}_i \leq a_i \leq a_{i_{\max}} - \tilde{a}_i \quad (i = 1, 2) \quad (43)
$$

$$
b_{\min} + \tilde{b} \leq b \leq b_{\max} - \tilde{b} \quad (44)
$$

$$
W_1 = F_1 \cos(\theta) - F_2 \cos(\beta) \quad (45)
$$

$$
W_2 = F_1 \sin(\theta) + F_2 \sin(\beta) \quad (46)
$$

$$
\theta = \arctan(L/b) \quad (47)
$$

$$
\beta = \arctan(L/(2L - b)) \quad (48)
$$

where the normal stress in each bar is given as $S_i$ and $F_i$ is the normal force applied to the $i$th bar.
The results for this case are shown in Table 2 and compared to the results obtained from the RDO formulation stemming from Truss Formulation 1. The results for deflection and volume are also shown in Fig. 5. Note that the solutions for RDO Truss Formulation 1 (circle) and RDO Truss Formulation 2 (star) are identical, which is further indication that certain equality constraints should be strictly satisfied regardless of all stochastic conditions.

We note that the obtained solutions are expected in that the overall performance for the deflection and volume has decreased from the deterministic to the non-deterministic problem. This is because minimizing the variation of deflection and volume comes at a cost of reduced performance. We also note that the dependencies in the equality constraints for Truss Formulation 2 (only L, b, W₁, and W₂ are independent) are a good indication that the constraints are Type 1 equality constraints.

5. Concluding Remarks

In this paper, we have presented an approach to solve the equality constrained design optimization problem under stochastic conditions. A taxonomy of optimization constraint types is developed, and an approach for handling each constraint under stochastic conditions is presented. As part of the development, we identify two types of equality constraints; those that must be strictly satisfied – regardless of all stochastic conditions, and those that cannot be strictly satisfied because of stochastic conditions. Mathematical definitions for each equality constraint type are provided, thus allowing for any given equality constraint to be classified. A simple structural optimization problem is used to show the developed approach.

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References


